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Recognizing locally equivalent graphs

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Abstract

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To locally complement a simple graph F at one of its vertices v is to replace the subgraph induced by F on $n(v)=\{w\colon vw$ is an edge of $F\}$ by the complementary subgraph. Graphs related by a sequence of local complementations are said to be locally equivalent. We describe invariants of locally equivalent graphs and a polynomial algorithm to recognize locally equivalent graphs. An application is given to counting the number of graphs locally equivalent to a given one.

1. Local complementations and applications

Let F be a simple graph. The *neighborhood* of a vertex v of F is $n(v)=\{w\colon vw$ is an edge of $F\}$. To *locally complement* F at v is to replace the subgraph induced by F on $n(v)$ by the complementary subgraph. We denote by $F*v$ the local complement of F at v, and we note that (i) $(F*v)*v=F$. A graph H is *locally equivalent* to F if it is obtained from F through successive local complementations. Equality (i) implies that it is actually an equivalence relation. We also note that any two locally equivalent graphs are defined over the same vertex set.

Local complementations are natural operations in the following applications.

(1) *Alternance (circle) graphs.* Let m be a word on a set of letters V and suppose that each letter occurs precisely twice in m. An *alternance* of m is a nonordered pair xy of letters such that we alternatively meet $\cdots x\cdots y\cdots x\cdots y\cdots$ or $\cdots y\cdots x\cdots y\cdots x\cdots$ when reading m. The simple graph on the vertex set V whose edges are the alternances of m is denoted by $A(m)$ and is called the *alternance graph* of m. Consider some $v\in V$ and let P, Q, R be the subwords of m such that $m=PvQvR$. Where $\tilde{Q}$ is the mirror

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image of Q and $m * v = Pv\tilde{Q}vR$, it is easy to verify that $A(m * v) = A(m) * v$. For example, if $m = 0410213243$ then $A(m)$ has edges $01, 12, 23, 34, 40$ and $m*1 = 0412013243$. This interpretation was introduced by Kotzig [15].

There is also a geometric interpretation. We consider finitely many chords of a circle and we suppose that no two chords have a common end. We label the ends of these chords in such a way that distinct ends get equal labels when they belong to the same chord. and only in this case. We run once along the circle and record the successive labels, determining in this way a word m where each label occurs precisely twice. Then two chords intersect if and only if the labels of their ends constitute an alternance of m. Alternance graphs are also called circle graphs, and the reader may refer to [13] for properties of circle graphs. Not every simple graph is an alternance graph. For example, the 5-wheel (the cycle of length 5 plus a central vertex joined to the vertices of this cycle) is not.

(2) *Splits and prime graphs.* Let G be a simple graph over the vertex set V. A bipartition $\{V', V''\}$ of V is called a *split* if $|V'|, |V''| \geqslant 2$ and there exist $W' \subseteq V'$ and $W'' \subseteq V''$ such that the cocycle of G between V' and V'' is equal to the edge set of the complete bipartite graph defined on the classes W' and W''. If G has no split, it is a *prime* graph. Splits and prime graphs were introduced by Cunningham [11]. It is easy to verify that any split is preserved after a local complementation (see [7] for details); consequently, prime graphs remain prime after a local complementation. The following result, proved in [4], shows that prime graphs and local complementations are strongly related. For a vertex v of G, we use the notation $G \backslash v$ for the subgraph induced by G over $V \backslash \{v\}$.

Reduction theorem. *Let G be a prime graph of order $\geqslant 6$, let v be a vertex of G, and let vw be an edge of G incident to v. Either $G \backslash v$ or $(G * v) \backslash v$ or $(G * v * w * v) \backslash v$ is prime.*

An interpretation of the three graphs $G \backslash v$, $(G * v) \backslash v$ and $(G * v * w * v) \backslash v$ is given in Section 7.

Based on the preceding theorem and the fact that any prime graph of order 5 is locally equivalent to a cycle of length 5, an efficient algorithm is given in [4] to recognize alternance graphs.

(3) *Distance-hereditary graphs.* A simple graph is said to be *distance-hereditary* if it is connected and the distance between any two vertices x and y is preserved when replacing G by any connected induced subgraph containing x and y. These graphs were introduced by Howorka [14] and characterized by Bandelt and Mulder [1]. It is proved in [6] that distance-hereditarity is preserved after a local complementation. In particular, a tree is clearly distance-hereditary, so that any graph locally equivalent to a tree is distance-hereditary. Those distance-hereditary graphs which are locally equivalent to trees are characterized in [6], and a polynomial algorithm to recognize these graphs is given.

2. Isotropic systems

Isotropic systems are algebraic and combinatorial structures introduced in [2, 5] with a view to unify some autodual properties of binary matroids and some properties of the Eulerian tours of 4-regular graphs. Another feature of isotropic systems is that they correspond naturally to classes of locally equivalent graphs. The properties stated in this section are proved in [5], except for (2.5), which is proved in [3].

For any finite set V, we consider $\mathscr{P}(V)$, the power set of V, with its canonical structure of vector space over GF(2). Thus, for $X, Y \subseteq V$, $X+Y$ is the symmetric difference of X and Y. The *neighborhood function* of a simple graph F over the vertex set V is the linear function $n: \mathscr{P}(V) \to \mathscr{P}(V)$ such that $n(v) = \{w: vw \text{ is an edge of } F\}$, $v \in V$.

Let K denote a 2-dimensional vector space over GF(2), provided with the bilinear form satisfying $\langle x, y\rangle = 1$ if and only if $0 \neq x \neq y \neq 0$. For any finite set V, we consider that the $2|V|$-dimensional vector space K^V is provided with the bilinear form $\langle A, B\rangle = \Sigma(\langle A(v), B(v)\rangle: v \in V)$. An *isotropic system* is a pair $S = (L, V)$, where V is a finite set and L is a totally isotropic subspace of K^V (i.e. $\langle A, B\rangle = 0$ for every $A, B \in L$) such that $\dim(L) = |V|$.

A vector $A \in K^V$ is said to be *complete* if $A(v) \neq 0$ for every $v \in V$. For $P \subseteq V$, let $AP \in K^V$ be defined by $AP(v) = A(v)$ if $v \in P$ and by $AP(v) = 0$ if $v \notin P$. Let $\hat{A} = \{AP: P \subseteq V\}$ and note that $\hat{A}$ is a subspace of K^V. If A is complete and $\dim(L \cap \hat{A}) = 0$ then A is called an *Eulerian vector* of S. The reader may refer to [5] for a correspondence between 4-regular graphs and isotropic systems such that Eulerian vectors correspond to Euler tours.

Two vectors $A, B \in K^V$ are *supplementary* if $0 \neq A(v) \neq B(v) \neq 0$ for every $v \in V$. A *graphic presentation* of an isotropic system $S = (L, V)$ is a triple (F, A, B), with a simple graph F over the vertex set V and two supplementary vectors $A, B \in K^V$ such that

$$L = \{An(P) + BP: P \subseteq V\},$$

where n is the neighborhood function of F. We call F a *fundamental graph* of S.

(2.1) *If (F, A, B) is a graphic presentation of an isotropic system S, then A is an Eulerian vector of S. Conversely, if A is an Eulerian vector of S, then there exists a graphic presentation (F', A', B') such that $A' = A$, and this graphic presentation is unique.*

(2.2) *Let A be an Eulerian vector of the isotropic system $S = (L, V)$, and let $v \in V$. There exists precisely one Eulerian vector A' of S satisfying $A'(v) \neq A(v)$ and $A'(w) = A(w)$ for every $w \in V \setminus \{v\}$.*

We use the notation $A * v$ to represent A' of (2.2). For any word $m = v_1 v_2 \cdots v_q$ on V, we let $A * m = (((A * v_1) * v_2) * \cdots) * v_q$.

(2.3) *If A and A' are any two Eulerian vectors of an isotropic system $S=(L, V)$, then there exists a word m on V such that $A' = A * m$.*

(2.4) *Let $P=(F, A, B)$ be a graphic presentation of an isotropic system $S=(L, V)$, and let $v \in V$. The graphic presentation of S induced by the Eulerian vector $A * v$ is $P * v = (F * v, A + Bv, B + An(v))$ (so that $A * v = A + Bv$).*

As a consequence of (2.3) and (2.4), the set of the fundamental graphs of a fixed isotropic system is a class of local equivalence. Property (2.1) says that any Eulerian vector determines precisely one fundamental graph, but, conversely, there may be more than one Eulerian vector associated with the same fundamental graph. The following property is proved in [3].

(2.5) *For any isotropic system S, there exists an integer k such that any fundamental graph F of S is associated with precisely k Eulerian vectors of S.*

We call the integer k of (2.5) the *index* of the isotropic system S. The index is used in Section 7 to count the number of graphs locally equivalent to a given graph F.

3. Some invariants of local equivalence

Throughout this section, F is a simple graph over the vertex set V, and n is the neighborhood function of F as defined in Section 2. For $X, Y \subseteq V$, $X + Y$ still represents the symmetric difference of X and Y.

For $X \subseteq V$, let $\pi = (\pi_{xy}: x \in X, y \in V \setminus X)$ be the matrix with coefficients in GF(2) such that $\pi_{xy} = 1$ if and only if xy is an edge of F. The function $c: X \to \operatorname{rank}(\pi)$, called the *connectivity function*, is introduced in [3]. It is proved that locally equivalent graphs have the same connectivity function. We conjectured [8] that any two graphs with the same connectivity function are locally equivalent. This has been disproved by Fon-Der-Flaass [12]. The two Petersen graphs depicted in Fig. 1 are not locally equivalent, whereas they have the same connectivity function.

The set $\Sigma = \{X \cup n(X): X \subseteq V\}$ is also invariant by local equivalence. Indeed, if we consider any isotropic system $S=(L, V)$ defined from a graphic presentation (F, A, B) with the given F as a fundamental graph, then Σ is the set of the supports of the vectors of L. A direct proof of the invariance of Σ is given by Fon-Der-Flaass in [12]. He also shows that the two graphs depicted in Fig. 1 give rise to the same set Σ, whereas they are not locally equivalent.

The following invariant is new. For $Y \subseteq V$, let $\mathrm{Odd}(F[Y])$ be the subset of the vertices with an odd degree in the induced subgraph $F[Y]$. For $X \subseteq V$, let $\mathrm{o}(X, F) = |\{Y: X = \mathrm{Odd}(F[Y])\}|$.

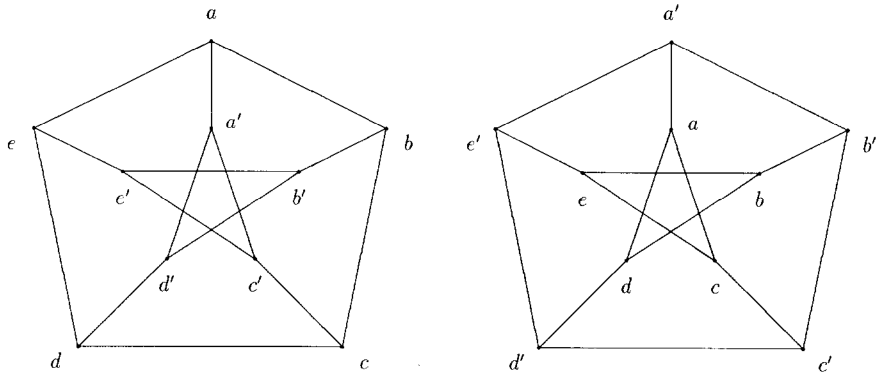

Fig. 1.

(3.1) *If F and F' are locally equivalent graphs on the vertex set V, then* $\mathrm{o}(X, F)=\mathrm{o}(X, F')$ *holds for every* $X \subseteq V$.

Proof. It is sufficient to verify the property for $F'=F*v$, $v\in V$. For $X, Y\subseteq V$, we let $\langle X, Y\rangle=|X\cap Y|$ (mod 2). Where n and n' are the neighborhood functions of F and F', respectively, the reader will first verify that

$$n'(P')=n(P')+\langle P', n(v)\rangle n(v)+P'\cap n(v), \quad P'\subseteq V.$$

Then, considering the map $P\rightarrow P'=P+v+\langle P, n(v)\rangle v$, which is an affine bijection from $\mathscr{P}(V)$ into $\mathscr{P}(V)$, the reader will verify that

$$\mathrm{Odd}(F[P'])=n'(P')\cap P'=n(P)\cap P=\mathrm{Odd}(F[P]),$$

which implies the statement. □

We do not know whether the family $(\mathrm{o}(X, F)\colon X\subseteq V)$ characterizes locally equivalent graphs. Anyway, this cannot give a good characterization. On the other hand, it is useful to prove that two graphs are not locally equivalent. For example, let us suppose that F is the cycle of length n, and let us consider two vertices x and y at distance d. Where f_p is the pth term of the Fibonacci sequence defined by $f_1=0$ and $f_2=1$, it is easy to verify that $\mathrm{o}(\{x, y\}, F)=f_d+f_{n-d}$. For $n>6$, the equality $f_1+f_{n-1}=f_p+f_{n-p}$ can only be satisfied for $p=1$. Therefore, two locally equivalent cycles of length $n>6$ must be equal. This no longer holds for $n\leqslant 6$.

Let E be the edge set of F and $\bar{E}$ the edge set of the complementary graph $\bar{F}$. By a (*topological*) *cycle* of F, we mean a subset $C\subseteq E$ such that the number of edges in C incident to any vertex of F is even. We recall that the set $\mathscr{Z}(F)$ of the cycles of F is a subspace of $\mathscr{P}(E)$. For $xy\in E\cup\bar{E}$, we let $v(xy)=n(x)\cap n(y)$, and, for any subset $Q\subseteq E\cup\bar{E}$, we let $v(Q)=\Sigma(v(xy)\colon xy\in Q)$. The *bineighborhood space* of F is the subspace of $\mathscr{P}(E)$ which is equal to the sum of the two subspaces $\{v(P)\colon P\in\mathscr{P}(\bar{E})\}$ and $\{v(C)\colon C\in\mathscr{Z}(F)\}$. We denote by $v(F)$ the bineighborhood space of F.

(3.2) *The bineighborhood spaces of two locally equivalent graphs have the same dimension.*

The preceding property may be obtained as a consequence of the study made in [10] to recognize locally equivalent graphs by an efficient algorithm. We give a direct proof in Section 5.

4. Recognizing efficiently locally equivalent graphs

We consider two simple graphs F_1 and F_2 over the same vertex set $V=\{1, 2, \ldots, n\}$. For $v, w, i \in V$, we define $\alpha_i^{vw}, \beta_i^{vw}, \gamma_i^{vw}, \delta_i^{vw} \in \mathrm{GF}(2)$ by

$$\alpha_i^{vw}=1 \Leftrightarrow iv \in E(F_1) \text{ and } iw \in E(F_2),$$

$$\beta_i^{vw}=1 \Leftrightarrow iv \in E(F_1) \text{ and } i=w,$$

$$\gamma_i^{vw}=1 \Leftrightarrow i=v \text{ and } iw \in E(F_2),$$

$$\delta_i^{vw}=1 \Leftrightarrow i=v=w.$$

The following results are proved in [10].

(4.1) *F_1 is locally equivalent to F_2 if and only if we can solve the following system of equations with $4n$ unknowns X_i, Y_i, Z_i, T_i in* GF(2), *$i \in V$:*

$$\sum_{i=1}^{i=n} (\alpha_i^{vw} Y_i + \beta_i^{vw} T_i + \gamma_i^{vw} X_i + \delta_i^{vw} Z_i) = 0 \quad \text{for every } v, w \in V, \tag{1}$$

$$X_i T_i + Y_i Z_i = 1 \quad \text{for every } i \in V. \tag{2}$$

The set of the solutions to (1) is a subspace $\mathscr{S}$ of $\mathrm{GF}(2)^{4n}$. By using a pivoting method, a base B of $\mathscr{S}$ can be computed in $O(n^4)$ time because there are $O(n^2)$ equations in (1). Then we can check each vector of $\mathscr{S}$ against condition (2) to find an eventual solution. Unfortunately, the dimension of $\mathscr{S}$ can be equal to $O(n)$, so that the enumeration of $\mathscr{S}$ is nonpolynomial in general. We prove in [10] that it is sufficient to enumerate a specified subset $\mathscr{S}' \subseteq \mathscr{S}$ satisfying $|\mathscr{S}'| = O(n^3)$ in order to find an eventual solution if it exists. The overall complexity of the algorithm is equal to $O(n^4)$.

5. Dimension of the bineighborhood space

In this section we consider $\nu(F)$, the bineighborhood space of a simple graph F, and we give a direct proof of (3.2). The neighborhood function of F is denoted by n. In general, we use the notation defined in Section 3.

For $P, Q \in \mathscr{P}(V)$, we let $\langle P, Q \rangle = |P \cap Q| \pmod 2$, and we note that $(P, Q) \to \langle P, Q \rangle$ is a bilinear form over the vector space $\mathscr{P}(V)$. For $P, X, Y \in \mathscr{P}(V)$, let $P*(X, Y) = P + \langle P, X \rangle Y$, and, for $N \subseteq \mathscr{P}(V)$, let $N*(X, Y) = \{P*(X, Y): P \in N\}$.

(5.1) *For a fixed pair* (X, Y), *the mapping* $\phi: P \to P*(X, Y)$ *is linear. If* $\langle X, Y\rangle = 0$ *then* $P*(X, Y)*(X, Y) = P$, *so that* ϕ *is a linear bijection from* $\mathscr{P}(V)$ *into* $\mathscr{P}(V)$ *and* $\dim(N*(X, Y)) = \dim(N)$ *holds for every subspace* N *of* $\mathscr{P}(V)$.

Proof. It is a simple verification. □

(5.2) *For any vertex* v *of* F, *we have*

$$\nu(F*v) = \nu(F)*(v, n(v)).$$

Proof. Let $F' = F*v$ and let n' be the neighborhood function of F'. We let $N = n(v) = n'(v)$ and we note that $n'(x) = n(x)$ if $x \notin N$ and $n'(x) = n(x) + N + x$ if $x \in N$. The linear bijection $\phi: P \to P*(v, n(v))$ is defined by $\phi(P) = P$ if $v \notin P$ and by $\phi(P) = P + N$ if $v \in P$. For $x, y \in V$, we let $\nu'(xy) = n'(x) \cap n'(y)$. We also use the notation $\nu'(F')$ instead of $\nu(F*v)$. To prove the property we consider some generating subset $\mathscr{G}$ of $\nu(F)$ and we verify that $\phi(G) \in \nu'(F')$ for every $G \in \mathscr{G}$.

We first consider $G = \nu(xy)$, $xy \in \bar{F}$. Due to the symmetric roles of x and y, we have three cases.

Case 1: $x = v$. We have $v \notin \nu(vy)$ because $v \notin n(v)$; so, $\phi(G) = \nu(vy)$. We also have $\nu(vy) = \nu'(vy)$ because $n'(y) = n(y)$. Therefore, $\phi(G) = \nu'(vy) \in \nu'(F')$.

Case 2: $x \neq v \neq y$, $y \notin N$. Then $v \notin \nu(xy)$ because $v \notin n(y)$, so that $\phi(G) = \nu(xy)$. We have $n'(y) = n(y)$ because $y \notin N$. If $x \notin N$ we also have $n'(x) = n(x)$, and we conclude as in case 1. If $x \in N$ we have $n'(x) = n(x) + N + x$, so that $\nu'(xy) = \nu(xy) + \nu(vy)$. But $\nu(vy) = \nu'(vy)$ as in case 1, so that $\phi(G) = \nu'(xy) + \nu'(vy) \in \nu'(F')$.

Case 3: $x \neq v \neq y$, $x, y \in N$. We have $v \in \nu(xy)$, so that $\phi(G) = \nu(xy) + N$. After locally complementing at v, xy becomes an edge of F', and we consider the triangle vxy of F'. We have

$$\begin{aligned}\nu'(vxy) &= n'(v) \cap n'(x) + n'(v) \cap n'(y) + n'(x) \cap n'(y)\\ &= N \cap (n(x) + N + x) + N \cap (n(y) + N + y) + (n(x) + N + x) \cap (n(y) + N + y)\\ &= \nu(xy) + N + n(x) \cap y + n(y) \cap x + x \cap y.\end{aligned}$$

The last three terms in the preceding sum are null because $x \neq y$ and xy is not an edge of F. Therefore, $\phi(G) = \nu'(vxy) \in \nu'(F')$.

We now consider $G = \nu(C)$, where C runs through a generating set $\mathscr{C}$ of $\mathscr{Z}(F)$. This subset $\mathscr{C}$ is the disjoint union of three sets $\mathscr{C}_1, \mathscr{C}_2$ and $\mathscr{C}_3$ defined as follows: $\mathscr{C}_1$ is the set of the triangles of F incident to v, $\mathscr{C}_2$ is the set of the elementary cycles of F incident to v which are not triangles and with no chord incident to v, $\mathscr{C}_3$ is the set of the elementary cycles of F incident to at most one vertex belonging to N. Any elementary cycle incident to v is expressible as a symmetric difference of cycles belonging to $\mathscr{C}_1 \cup \mathscr{C}_2$, and this is also true for every elementary cycle incident to $\geqslant 2$ vertices belonging to N. Thus, any elementary cycle is generated by $\mathscr{C}$, which implies that $\mathscr{C}$ is a generating set of $\mathscr{Z}(F)$.

Case (i): $C\in\mathscr{C}_1$. We verify that $v\in\nu(C)$, so that $\phi(G)=\nu(C)+N$. Where x and y are the two other vertices incident to C, case 3 yields $\nu(C)=\nu(vxy)=\nu'(xy)+N$ when exchanging the roles of F and F'. Therefore, $\phi(G)=\nu'(xy)\in\nu'(F')$.

Case (ii): $C\in\mathscr{C}_2$. We have $v\notin\nu(C)$ because the length of C is $\geqslant 4$, so that $\phi(G)=\nu(C)$. Moreover, C is still a cycle of F', and we consider $\nu'(C)$. Let zx, xv, vy, yt be the edges of C which constitute a path of length 4 centered at v. For every vertex u incident to C and not belonging to $\{x, v, y\}$, we have $n'(u)=n(u)$. Therefore, $\nu'(e)=\nu(e)$ for every edge of C not belonging to $\{zx, xv, vy, yt\}$. We verify that

$$\nu'(zx)=\nu(zx)+\nu(vz)+x=\nu(zx)+\nu'(vz)+x,$$

$$\nu'(xv)=\nu(xv)+N+x,$$

$$\nu'(vy)=\nu(vy)+N+y,$$

$$\nu'(yt)=\nu(yt)+\nu(vt)+y=\nu(yt)+\nu'(vt)+y.$$

This implies

$$\nu'(C)=\nu(C)+\nu'(vz)+\nu'(vt),$$

so that

$$\phi(G)=\nu'(C)+\nu'(vz)+\nu'(vt)\in\nu'(F').$$

Case (iii): $C\in\mathscr{C}_3$. We verify that $v\notin\nu(C)$, so that $\phi(G)=\nu(C)$. The cycle C still exists in F', and we consider $\nu'(C)$. We have $\nu'(e)=\nu(e)$ for every edge e of C, with the exception of two eventual edges, say xy and xz, incident to a vertex $x\in N$. We verify that $\nu'(xy)=\nu(xy)+\nu'(vy)$ and $\nu'(xz)=\nu(xz)+\nu'(vz)$. Therefore, we have either $\phi(G)=\nu(C)$ or $\phi(G)=\nu(C)+\nu'(vy)+\nu'(vz)$, and in both cases $\phi(G)\in\nu'(F')$.

$\mathscr{G}=\{\nu(xy)\colon xy\in\bar{F}\}\cup\{\nu(C)\colon C=\mathscr{C}\}$ is a generating subset of $\nu(F)$ and, so, the proof is complete. $\square$

Proof of (3.2). It is sufficient to prove the result for a pair of graphs $\{F, F'\}$, with $F'=F*v$ for some vertex v. Following (5.2) and (5.1), we have $\dim(\nu(F*v))=\dim(\nu(F))$ because $\langle v, n(v)\rangle=0$. $\square$

6. Ends

We still consider the graph F and we use the notation of Section 3. Two vetices x and y are called *twins* if either $n(x)=n(y)$ – *unrelated twins* – or xy is an edge and $n(x)-y=n(y)-x$ – *related twins*. An *end* of F is either a pendent vertex or a pair of twins. We say that some $P\in\mathscr{P}(V)$ is *orthogonal* to a subspace S of $\mathscr{P}(V)$ if $\langle P, Q\rangle=0$ for every $Q\in S$. We let $S^{\perp}=\{P\in\mathscr{P}(V)\colon P \text{ is orthogonal to } S\}$.

(6.1) *Any end of F is orthogonal to $\nu(F)$.*

Proof. We successively consider the three kinds of ends.

Case 1: Let s be a pendent vertex of F. We have $s \notin v(xy)$ for any $xy \in E \cup \bar{E}$ since, otherwise, s would have a degree $\geqslant 2$. Therefore, $s \notin v(P+C)$ for any $P \in \mathscr{P}(\bar{E})$ and any $C \in \mathscr{X}(F)$ and, so, s is orthogonal to $v(F)$.

Case 2: Let $\{s, v\}$ be a pair of related twins of F and let $Q \in v(F)$. We have to verify that $\langle \{s, v\}, Q \rangle = \langle s+v, Q \rangle = 0$. By (4.3), after exchanging the roles of F and $F * v$, there exists $P \in v(F * v)$ such that $Q = P + \langle v, P \rangle n(v)$. We easily verify that s is a pendant vertex of $F * v$ and, so, case 1 implies $\langle s, P \rangle = 0$. We also have $\langle v, n(v) \rangle = 0$ and $\langle s, n(v) \rangle = 1$. Therefore,

$$
\begin{aligned}
\langle s+v, Q \rangle &= \langle s+v, P + \langle v, P \rangle n(v) \rangle \\
&= \langle s, P \rangle + \langle v, P \rangle + \langle v, P \rangle \langle s, n(v) \rangle + \langle v, P \rangle \langle v, n(v) \rangle \\
&= \langle v, P \rangle + \langle v, P \rangle = 0.
\end{aligned}
$$

Case 3: Let $\{s, t\}$ be a pair of unrelated twins of F and let $Q \in v(F)$. We have to verify that $\langle \{s, t\}, Q \rangle = \langle s+t, Q \rangle = 0$. Let v be a common neighbor of s and t. By (4.3), after exchanging the roles of F and $F * v$, there exists $P \in v(F * v)$ such that $Q = P + \langle v, P \rangle n(v)$. We easily verify that $\{s, t\}$ is a pair of unrelated twins of $F * v$ and, so, case 2 implies $\langle s+t, P \rangle = 0$. We also have $\langle s+t, n(v) \rangle = 0$ because v is a common neighbor to s and t. Therefore,

$$
\begin{aligned}
\langle s+t, Q \rangle &= \langle s+t, P + \langle v, P \rangle n(v) \rangle \\
&= \langle s+t, P \rangle + \langle v, P \rangle \langle s+t, n(v) \rangle = 0. \qquad \square
\end{aligned}
$$

Thus, if we consider the subspace $\varepsilon(F)$ generated by the ends of F, we have $\varepsilon(F) \subseteq v(F)^{\perp}$. Equality does not hold in general. If we consider $F = K_{n,n} - nK_2$ (the complete bipartite graph minus a perfect matching) for an even integer n, then $v(F)^{\perp}$ has dimension 2 (the chromatic classes belong to $v(F)^{\perp}$) when $\varepsilon(F)$ has dimension 0.

7. Enumerating a class of local equivalence

Let F be a simple graph over the vertex set V. We want to compute $l(F)$, the number of graphs locally equivalent to F. Let S be an isotropic system which admits F as a fundamental graph, and let $e(S)$ be the number of Eulerian vectors of S. Then, following (2.5), we have

$$l(F) = e(S)/k(S)$$

where $k(S)$ is the index of S.

We say that F is in *class* μ if it satisfies the following three properties: (i) every vertex of F has an odd degree; (ii) $|v(xy)|$ is even for every edge xy of $\bar{F}$; and

(iii) $|v(C)|=|C|\,(\mathrm{mod}\,2)$ for every cycle C of F. Where $v(F)^{\perp}$ is the orthogonal subspace of $v(F)$ in $\mathscr{P}(V)$, it is proved in [14] that

$$k(S)=\begin{cases}|v(F)^{\perp}|+2 & \text{if } F \text{ is in class } \mu,\\ |v(F)^{\perp}| & \text{otherwise.}\end{cases} \tag{3}$$

Let $K'=K\setminus\{0\}$. The *Tutte–Martin polynomial* of S is

$$M(S;\zeta)=\Sigma((\zeta-2)^{\dim(L\cap\hat{A})}\colon A\in K'), \tag{4}$$

so that

$$e(S)=M(S;2).$$

For $x\in K'$ and $v\in V$, let

$$L_x^v=\{A\in L\colon A(v)=x\},$$

$$L|_x^v=\text{projection of } L_x^v \text{ over } K^{V-x},$$

$$S|_x^v=(L|_x^v,\ V-x).$$

It is proved in [5] that $S|_x^v$ is an isotropic system, and it is called an *elementary minor* of S at v. Let $K'=\{x,y,z\}$. Then there are three elementary minors at v, which are $S|_x^v$, $S|_y^v$, $S|_z^v$. If v is isolated in F (which implies that v is isolated in every fundamental graph of S) then the three elementary minors are equal to the same isotropic system which we denote by $S|_\cdot^v$. We have the following formulas, proved in [9], to compute recursively $M(S;\zeta)$:

$$M(S;\zeta)=\begin{cases}\zeta M(S|_\cdot^v;\zeta) & \text{if } v \text{ is isolated},\\ M(S|_x^v;\zeta)+M(S|_y^v;\zeta)+M(S|_z^v;\zeta) & \text{otherwise.}\end{cases} \tag{5}$$

To compute $M(S;\zeta)$, it is not necessary to actually consider S, but only a fundamental graph F of S. Let us define

$$M(F;\zeta)=M(S;\zeta).$$

This definition is consistent because if S' is another isotropic system which admits F as a fundamental graph, then S is isomorphic to S', and (4) clearly implies $M(S;\zeta)=M(S';\zeta)$. More generally, this also holds if S' is an isotropic system which admits a fundamental graph F' locally equivalent to F. Therefore,

$$M(F;\zeta)=M(F';\zeta)\quad \text{if } F' \text{ is locally equivalent to } F. \tag{6}$$

It is proved in [5] that if v is isolated in F, then $F\setminus v$ is a fundamental graph of $S|_\cdot^v$. Otherwise, consider any edge vw incident to v. Then $F\setminus v$, $F*v\setminus v$, $F*vwv\setminus v$ are, up to the ordering, the fundamental graphs of $S|_x^v$, $S|_y^v$, $S|_z^v$. Therefore, (5) implies

$$M(F;\zeta)=\begin{cases}\zeta M(F\setminus v;\zeta) & \text{if } v \text{ is isolated},\\ M(F\setminus v;\zeta)+M(F*v\setminus v;\zeta)+M(F*vwv\setminus v;\zeta) & \text{otherwise.}\end{cases} \tag{7}$$

If we let $e(F)=e(S)$, then the following formulas for computing $e(F)$ follow:

$$e(F)=\begin{cases}2e(F\setminus v) & \text{if } v \text{ is isolated,}\\ e(F\setminus v)+e(F*v\setminus v)+e(F*vwv\setminus v) & \text{otherwise.}\end{cases} \tag{8}$$

Finally, the recursive computation is initialized with $e(F)=2$ if $|V|=1$.

For example, let us consider P_n, the graph which is a path with n vertices, $n\geqslant 1$, and let $p_n=e(P_n)$. Taking v as an endpoint and choosing vw in the only possible way, we verify that $P_n\setminus v=P_n*v\setminus v=P_{n-1}$ and $P_n*vwv\setminus v=P_{n-2}\dot{\cup}P_1$, where $\dot{\cup}$ stands for the disjoint union. Since P_1 is an isolated vertex, we get

$$\begin{aligned}P_n&=2e(P_{n-1})+e(P_1\dot{\cup}P_{n-2})\\ &=2p_{n-1}+2p_{n-2},\end{aligned}$$

with $p_1=2$ and $p_2=6$. The solution of this linear recurrence relation is

$$p_n=\sqrt{3}((1+\sqrt{3})^{n+1}-(1-\sqrt{3})^{n+1})/6.$$

To compute $l(P_n)$ we need to know $k(P_n)$. Since P_n is not a graph of class μ, we have $k(P_n)=2|v(P_n)^{\perp}|$. Let us denote by $x_1, x_2, \ldots, x_n$ the successive vertices of P_n. We have $v(x_ix_{i+2})=x_{i+1}$ for $1\leqslant i\leqslant n-2$, so that $\dim(v(F)^{\perp})\leqslant 2$. Following the results of Section 6, equality holds because P_n has two endpoints. Thus,

$$l(P_n)=p_n/4.$$

For another example, let us consider C_n, the cycle of length $n\geqslant 5$. We observe that C_3 and C_4 are locally equivalent to P_3 and P_4, so that $l(C_3)=l(P_3)$ and $l(C_4)=l(P_4)$. Taking an arbitrary edge vw on C_n, we easily verify that

$$\begin{aligned}&C_n\setminus v=P_{n-1},\\ &C_n*v\setminus v=C_{n-1},\\ &C_n*vwv\setminus v=C'_{n-2},\end{aligned}$$

where C'_{n-2} denotes the result of adding a pendent vertex to a cycle of length $n-2$. Taking v as the pendent vertex of C'_{n-2} and choosing w in the only possible way, we verify that

$$\begin{aligned}&C'_{n-2}\setminus v=C'_{n-2}*v\setminus v=C_{n-2},\\ &C'_{n-2}*vwv\setminus v=P_{n-3}\dot{\cup}P_1.\end{aligned}$$

Where $c_n=e(C_n)$, Eq. (8) implies

$$c_n=c_{n-1}+2c_{n-2}+P_{n-1}+2p_{n-3}.$$

The solution of this linear recurrence relation yields

$$c_n=(1+\sqrt{3})^n+(1-\sqrt{3})^n-4(2^{n-1}+(-1)^n)/3.$$

It is proved in [10] that any graph G of girth $\geqslant 5$ satisfies $k(G)=1$. So, we have $k(C_n)=1$ when $n \geqslant 5$ (which is easy to prove directly) and

$$l(C_n)=c_n.$$

As a last example, let us consider W_5, the 5-wheel constructed by adding a central vertex v to C_5 and joining v to every vertex of C_5. Where w is any vertex of C_5, we easily verify that $W_5 * v$ and $W_5 * vwv$ are isomorphic to W_5, with v still the central vertex in the first case whereas w is the central vertex in the second case. Thus, $W_5 \backslash v$ and $W_5 * v \backslash v$ are isomorphic to C_5 whereas $W_5 * vwv \backslash v$ is isomorphic to C_5 plus two chords incident to w. It is easy to verify that the last graph is locally equivalent to C_5. Therefore, (8) implies

$$e(W_5)=3e(C_5).$$

We verify that W_5 belongs to class μ and that $v(W_5)^{\perp}$ has dimension 0. Therefore, $k(W_5)=3$, so that

$$l(W_5)=e(C_5)=l(C_5)=c_5.$$

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